Correction Exercices 1 et 6 du DS n° 3

Exercice 1.

❀❀ **1.** $S_1 = 2\sum_{k=0}^{17} k + \sum_{k=0}^{17} 7$
$= 2 \times \frac{17\times 18}{2} - 7 \times (17+1)$
$= 17 \times 18 - 7 \times 18$
$= 10 \times 18$
$\boxed{S_1 = 180}$

$S_2 = \sum_{p=3}^{12} \frac{1}{2^p} + \sum_{p=3}^{12} p$
$= \sum_{p=3}^{12} \left(\frac{1}{2}\right)^p + \sum_{p=3}^{12} p$
$= \left(\frac{1}{2}\right)^3 \frac{1-(\frac{1}{2})^{12-3+1}}{1-\frac{1}{2}} + (12-3+1)\frac{12+3}{2}$
$= \frac{1}{8}\frac{1-\frac{1}{1024}}{\frac{1}{2}} + 10 \times \frac{15}{2}$
$= \frac{1}{4} \times \frac{1023}{1024} + 75$
$\boxed{S_2 = \frac{1023}{4096} + 75}$

❀ **2.** **(a)** $\frac{1}{2(k-1)} - \frac{1}{2(k+1)} = \frac{k+1-(k-1)}{2(k-1)(k+1)} = \frac{2}{2(k^2-1)} = \frac{1}{k^2-1}$, donc $\boxed{\forall k \in [\![2,+\infty[\![,\ \frac{1}{2(k-1)} - \frac{1}{2(k+1)} = \frac{1}{k^2-1}}$.

(b) Pour $n \geqslant 2$, $\sum_{k=2}^{n} \frac{1}{k^2-1} = \frac{1}{2}\left(\sum_{k=2}^{n}\frac{1}{k-1} - \sum_{k=2}^{n}\frac{1}{k+1}\right)$

$= \frac{1}{2}\left(\sum_{p=1}^{n-1}\frac{1}{p} - \sum_{\ell=3}^{n+1}\frac{1}{\ell}\right)$ avec $p = k-1$: k va de 2 à n, p va de 1 à $n-1$; et $\ell = k+1$: k va de 2 à n, ℓ va de 3 à $n+1$

$= \frac{1}{2}\left(\frac{1}{1} + \frac{1}{2} + \sum_{p=3}^{n-1}\frac{1}{p} - \sum_{\ell=3}^{n-1}\frac{1}{\ell} - \frac{1}{n} - \frac{1}{n+1}\right)$

$$\boxed{\sum_{k=2}^{n} \frac{1}{k^2-1} = \frac{1}{2}\left(\frac{3}{2} - \frac{2n+1}{n^2+n}\right)}$$

❀❀ **3.** **(a)** $A(x) = x^6 + 6x^5 \times 2 + 10x^4 \times 2^2 + 20x^3 \times 2^3 + 10x^2 \times 2^4 + 6x \times 2^5 + 2^6$
$\boxed{A(x) = x^6 + 12x^5 + 60x^4 + 160x^3 + 240x^2 + 192x + 64}$

(b) $(1-2i)^5 = 1 + 5\times(-2i) + 10\times(-2i)^2 + 10\times(-2i)^3 + 5\times(-2i)^4 + (-2i)^5$
$= 1 - 10i + 10\times 4i^2 - 10\times 8i^3 + 5\times 16i^4 - 32i^5$
$= 1 - 10i - 40 + 80i + 80 - 32i$
$\boxed{(1-2i)^5 = 41 + 38i}$

$$\sum_{k=0}^{4}(1-2i)^k = \frac{1-(1-2i)^5}{1-(1-2i)} = \frac{1-(41+38i)}{2i} = \frac{(-40-38i)\times(-i)}{2} = \boxed{-19+20i}$$

❀❀ **4.** **(a)** $\tan(x) = \frac{1}{\sqrt{3}} \iff \exists k \in \mathbb{Z},\ x = \arctan\left(\frac{1}{\sqrt{3}}\right) + k\pi$
$\iff \exists k \in \mathbb{Z},\ x = \frac{\pi}{6} + k\pi$

Ainsi $\boxed{\mathcal{S}_{\mathbb{R}} = \left\{\frac{\pi}{6} + k\pi,\ k \in \mathbb{Z}\right\}}$, et $\boxed{\mathcal{S}_{[0,2\pi[} = \left\{\frac{\pi}{6}, \frac{7\pi}{6}\right\}}$.

(b) $\cos(3x + \frac{\pi}{6}) = \frac{\sqrt{2}}{2} \iff \exists k \in \mathbb{Z},\quad 3x + \frac{\pi}{6} = \frac{\pi}{4} + 2k\pi$ ou $3x + \frac{\pi}{6} = -\frac{\pi}{4} + 2k\pi$
$\iff \exists k \in \mathbb{Z},\quad 3x = \frac{\pi}{4} - \frac{\pi}{6} + 2k\pi$ ou $3x = -\frac{\pi}{4} - \frac{\pi}{6} + 2k\pi$
$\iff \exists k \in \mathbb{Z},\quad x = \frac{\pi}{36} + \frac{2k\pi}{3}$ ou $x = \frac{-5\pi}{36} + \frac{2k\pi}{3}$

Ainsi $\boxed{\mathcal{S}_{\mathbb{R}} = \left\{\frac{\pi}{36} + \frac{2k\pi}{3},\ k \in \mathbb{Z}\right\} \cup \left\{\frac{-5\pi}{36} + \frac{2k\pi}{3},\ k \in \mathbb{Z}\right\}}$.

Et $\boxed{\mathcal{S}_{[0,2\pi[} = \left\{\frac{\pi}{36}; \frac{25\pi}{36}; \frac{49\pi}{36}; \frac{19\pi}{36}; \frac{43\pi}{36}; \frac{67\pi}{36}\right\}}$.

(c) $$\begin{aligned} \sin(x) - \sin(3x) = 0 &\iff \sin(x) = \sin(3x) \\ &\iff \exists k \in \mathbb{Z},\quad x = 3x + 2k\pi \ \text{ ou } \ x = \pi - 3x + 2k\pi \\ &\iff \exists k \in \mathbb{Z},\quad -2x = 2k\pi \text{ ou } 4x = \pi + 2k\pi \\ &\iff \exists k \in \mathbb{Z},\quad x = -k\pi \ \text{ ou } \ x = \frac{\pi}{4} + \frac{k\pi}{2} \end{aligned}$$

Ainsi $\boxed{\mathcal{S}_{\mathbb{R}} = \{-k\pi,\ k \in \mathbb{Z}\} \cup \left\{\frac{\pi}{4} + \frac{k\pi}{2},\ k \in \mathbb{Z}\right\}}$ et $\boxed{\mathcal{S}_{[0,2\pi[} = \left\{0; \pi; \frac{\pi}{4}; \frac{3\pi}{4}; \frac{5\pi}{4}; \frac{7\pi}{4}\right\}}$.

5. (a) $|2x+3| \leqslant 1 \iff -1 \leqslant 2x+3 \leqslant 1 \iff -4 \leqslant 2x \leqslant -2 \iff -2 \leqslant x \leqslant -1$

$\boxed{\mathcal{S} = [-2; -1]}$.

(b) $|-x+1| = \begin{cases} -(-x+1) & \text{si } -x+1 \leqslant 0 \\ -x+1 & \text{si } -x+1 > 0 \end{cases}$ c'est-à-dire $|-x+1| = \begin{cases} x-1 & \text{si } x > 1 \\ -x+1 & \text{si } x \leqslant 1 \end{cases}$

De même, $|2x-6| = \begin{cases} -2x+6 & \text{si } x \leqslant 3 \\ 2x-6 & \text{si } x > 3 \end{cases}$

Donc :

x		1		3	
$\lvert -x+1\rvert$	$-x+1$	|	$x-1$	|	$x-1$
$\lvert 2x+6\rvert$	$-2x+6$	|	$-2x+6$	|	$2x-6$
$\lvert -x+1\rvert + \lvert 2x+6\rvert = 3 \iff$	$-3x+7=3$	|	$-x+5=3$	|	$3x-7=3$
	$-3x=-4$	|	$-x=-2$	|	$3x=10$
	$x=\frac{4}{3}$	|	$x=2$	|	$x=\frac{10}{3}$
	$\frac{4}{3} \notin]-\infty, 1]$	|	$2 \in [1;3]$	|	$\frac{10}{3} > 3$

Finalement, $\boxed{\mathcal{S} = \left\{2; \frac{10}{3}\right\}}$.

Exercice 6.

1. (a) $$\begin{aligned} 1 + e^{i\frac{\pi}{3}} &= 1 + \cos(\tfrac{\pi}{3}) + i\sin(\tfrac{\pi}{3}) \\ &= 1 + \tfrac{1}{2} + i\tfrac{\sqrt{3}}{2} \\ &= \tfrac{3}{2} + i\tfrac{\sqrt{3}}{2} \end{aligned}$$ et $$\begin{aligned} \sqrt{3}e^{i\frac{\pi}{6}} &= \sqrt{3}\cos(\tfrac{\pi}{6}) + i\sqrt{3}\sin(\tfrac{\pi}{6}) \\ &= \sqrt{3} \times \tfrac{\sqrt{3}}{2} + i\sqrt{3}\tfrac{1}{2} \\ &= \tfrac{3}{2} + i\tfrac{\sqrt{3}}{2} \end{aligned}$$

Donc $\boxed{1 + e^{i\frac{\pi}{3}} = \sqrt{3}e^{i\frac{\pi}{6}}}$.

(b) $e^{i(\frac{k\pi}{3} + \frac{\pi}{4})} = \cos(\frac{k\pi}{3} + \frac{\pi}{4}) + i\sin(\frac{k\pi}{3} + \frac{\pi}{4})$.

2. (a) $$\begin{aligned} P_n &= \sum_{k=0}^{n} \binom{n}{k} \cos\left(\frac{k\pi}{3} + \frac{\pi}{4}\right) + i\sum_{k=0}^{n} \binom{n}{k} \sin\left(\frac{k\pi}{3} + \frac{\pi}{4}\right) \\ &= \sum_{k=0}^{n} \binom{n}{k} \left(\cos\left(\frac{k\pi}{3} + \frac{\pi}{4}\right) + i\sin\left(\frac{k\pi}{3} + \frac{\pi}{4}\right)\right) \\ &= \sum_{k=0}^{n} \binom{n}{k} e^{i(\frac{k\pi}{3} + \frac{\pi}{4})} \\ &= e^{i\frac{\pi}{4}} \sum_{k=0}^{n} \binom{n}{k} \left(e^{i\frac{\pi}{3}}\right)^k \\ &= e^{i\frac{\pi}{4}} \sum_{k=0}^{n} \binom{n}{k} \left(e^{i\frac{\pi}{3}}\right)^k \times 1^{n-k} \\ &= e^{i\frac{\pi}{4}} \left(1 + e^{i\frac{\pi}{3}}\right)^n \qquad \text{par binôme de Newton} \\ &= e^{i\frac{\pi}{4}} \left(\sqrt{3}e^{i\frac{\pi}{6}}\right)^n \qquad \text{d'après } \mathbf{1.(a)} \end{aligned}$$

$\boxed{P_n = e^{i\frac{\pi}{4}} \times (\sqrt{3})^n \times e^{i\frac{n\pi}{6}}}$

(b) $e^{i\frac{\pi}{4}} = \cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}(1+i)$.

Donc $P_n = (\sqrt{3})^n \times \frac{\sqrt{2}}{2}(1+i)\left(\cos\left(\frac{n\pi}{6}\right) + i\sin\left(\frac{n\pi}{6}\right)\right)$

$$= (\sqrt{3})^n \times \frac{\sqrt{2}}{2}\left(\cos\left(\frac{n\pi}{6}\right) + i\sin\left(\frac{n\pi}{6}\right) + i\cos\left(\frac{n\pi}{6}\right) + i^2\sin\left(\frac{n\pi}{6}\right)\right)$$

$$= \frac{\sqrt{2}\times(\sqrt{3})^n}{2}\left(\cos\left(\frac{n\pi}{6}\right) - \sin\left(\frac{n\pi}{6}\right) + i\left(\sin\left(\frac{n\pi}{6}\right) + \cos\left(\frac{n\pi}{6}\right)\right)\right)$$

Or $C_n = \mathrm{Re}\,(P_n)$ donc $\boxed{C_n = \frac{\sqrt{2}\times(\sqrt{3})^n}{2}\left(\cos\left(\frac{n\pi}{6}\right) - \sin\left(\frac{n\pi}{6}\right)\right)}$.

(c) $S_n = \mathrm{Im}\,(P_n)$ donc $\boxed{S_n = \frac{\sqrt{2}\times(\sqrt{3})^n}{2}\left(\cos\left(\frac{n\pi}{6}\right) + \sin\left(\frac{n\pi}{6}\right)\right)}$.

Bonus :

(a) Pour $n \in \mathbb{N}^*$, $C_n = 0 \iff \cos\left(\frac{n\pi}{6}\right) = \sin\left(\frac{n\pi}{6}\right)$

$$\iff \cos\left(\frac{n\pi}{6}\right) = \cos\left(\frac{\pi}{2} - \frac{n\pi}{6}\right)$$

$$\iff \frac{n\pi}{6} = \frac{\pi}{2} - \frac{n\pi}{6}\ (2\pi) \text{ ou } \frac{n\pi}{6} = -\frac{\pi}{2} + \frac{n\pi}{6}\ (2\pi)$$

$$\iff \frac{n\pi}{3} = \frac{\pi}{2}\ (2\pi) \text{ ou } \frac{\pi}{2} = 0\ (2\pi)$$

Or $\frac{n\pi}{3} = \frac{\pi}{2}\ (2\pi) \iff \exists k \in \mathbb{Z}, n = \frac{3}{2} + 6k$ ce qui n'est pas possible car n et k sont entiers.

Et $\frac{\pi}{2} = 0\ (2\pi)$ n'est pas possible non plus car $\frac{\pi}{2}$ et 0 sont tous deux dans l'intervalle $[0, 2\pi]$.

Donc $\boxed{C_n \text{ n'est jamais nul}}$.

(b)

$$Q_n = \frac{\frac{\sqrt{2}\times(\sqrt{3})^n}{2}\left(\cos\left(\frac{n\pi}{6}\right) + \sin\left(\frac{n\pi}{6}\right)\right)}{\frac{\sqrt{2}\times(\sqrt{3})^n}{2}\left(\cos\left(\frac{n\pi}{6}\right) - \sin\left(\frac{n\pi}{6}\right)\right)}$$

$$= \frac{\frac{\sqrt{2}}{2}\cos\left(\frac{n\pi}{6}\right) + \frac{\sqrt{2}}{2}\sin\left(\frac{n\pi}{6}\right)}{\frac{\sqrt{2}}{2}\cos\left(\frac{n\pi}{6}\right) - \frac{\sqrt{2}}{2}\sin\left(\frac{n\pi}{6}\right)}$$

$$= \frac{\cos(\frac{\pi}{4})\cos\left(\frac{n\pi}{6}\right) + \sin(\frac{\pi}{4})\sin\left(\frac{n\pi}{6}\right)}{\sin(\frac{\pi}{4})\cos\left(\frac{n\pi}{6}\right) - \cos(\frac{\pi}{4})\sin\left(\frac{n\pi}{6}\right)}$$

$$= \frac{\sin\left(\frac{\pi}{4} + \frac{n\pi}{6}\right)}{\cos\left(\frac{\pi}{4} + \frac{n\pi}{6}\right)}$$

$$\boxed{Q_n = \tan\left(\frac{\pi}{4} + \frac{n\pi}{6}\right)}$$