Corrigé du DM n° 9

Correction 1.

$$S_1 = 3\sum_{k=0}^{7}(k-2^k) = 3\sum_{k=0}^{7}k - 3\sum_{k=0}^{7}2^k = 3\frac{7\times 8}{2} - 3\frac{1-2^8}{1-2} = 3\times 7\times 4 + 3 - 3\times 2^8 = 84+3-768 = \boxed{-681}$$

$$S_2 = \frac{1}{6}\sum_{k=8}^{31}k - \frac{1}{6}\sum_{k=8}^{31}5 = \frac{1}{6}(31-8+1)\frac{31+8}{2} - \frac{1}{6}\times 5\times(31-8+1) = 4\times\frac{39}{2} - 4\times 5 = \boxed{58}$$

$$S_3(n) = \sum_{k=0}^{n}2^k + 4\sum_{k=0}^{n}k + \sum_{k=0}^{n}(n-3) = \frac{1-2^{n+1}}{1-2} + 4\frac{n(n+1)}{2} + (n+1)(n-3) = 2^{n+1} - 1 + 2n(n+1) + (n+1)(n-3) = 2^{n+1} + (n+1)(3n-3) - 1 = \boxed{2^{n+1}+3n^2-4}$$

$$S_4(n) = \sum_{k=0}^{n}2^k\times 3^n\times 3^{-k} = 3^n\sum_{k=0}^{n}\left(\frac{2}{3}\right)^k = 3^n\frac{1-\left(\frac{2}{3}\right)^{n+1}}{1-\frac{2}{3}} = 3^n\times 3\times\left(1-\left(\frac{2}{3}\right)^{n+1}\right) = 3^{n+1}\times\left(1-\left(\frac{2}{3}\right)^{n+1}\right) = \boxed{3^{n+1}-2^{n+1}}$$

Correction 2.

1. $$\frac{a}{k+1}+\frac{b}{k+3} = \frac{a(k+3)}{(k+1)(k+3)} + \frac{b(k+1)}{(k+1)(k+3)} = \frac{ak+3a+bk+b}{(k+1)(k+3)} = \frac{(a+b)k+3a+b}{(k+1)(k+3)}$$

On cherche a et b tels que $a+b=0$ et $3a+b=1$.

Donc $a=-b$ et $-2b=1$ soit $\boxed{b=-\frac{1}{2} \text{ et } a=\frac{1}{2}}$.

Donc $S_n = \sum_{k=0}^{n}\left(\frac{1}{2(k+1)} - \frac{1}{2(k+3)}\right) = \frac{1}{2}\sum_{k=0}^{n}\frac{1}{k+1} - \frac{1}{2}\sum_{k=1}^{n}\frac{1}{k+3}$

Dans la première somme, on pose $p=k+1$, k va de 0 à n, donc p va de 1 à $n+1$ et dans la 2ème somme, on pose $q=k+3$, k va de 0 à n, donc q va de 3 à $n+3$

Donc $$S_n = \frac{1}{2}\sum_{p=1}^{n+1}\frac{1}{p} - \frac{1}{2}\sum_{q=3}^{n+3}\frac{1}{q} = \frac{1}{2}\left(\frac{1}{1}+\frac{1}{2}+\sum_{p=3}^{n+1}\frac{1}{p} - \sum_{q=3}^{n+1}\frac{1}{q} - \frac{1}{n+2} - \frac{1}{n+3}\right) = \boxed{\frac{1}{2}\left(\frac{3}{2}-\frac{1}{n+2}-\frac{1}{n+3}\right)}$$

*** 2. (grandes étapes seulement).**

On cherche a, b et c tels que $\frac{1}{k(k+1)(k+2)} = \frac{a}{k} + \frac{b}{k+1} + \frac{c}{k+2}$.

On trouve $a = c = \frac{1}{2}$ et $b = -1$, ainsi $\displaystyle \frac{1}{k(k+1)(k+2)} = \frac{1}{2k} - \frac{1}{k+1} + \frac{1}{2(k+2)}$.

$$\begin{aligned}
\text{Donc} \quad T_n &= \frac{1}{2}\sum_{k=1}^{n}\frac{1}{k} - \sum_{k=1}^{n}\frac{1}{k+1} + \frac{1}{2}\sum_{k=1}^{n}\frac{1}{k+2} \\
&= \frac{1}{2}\sum_{k=1}^{n}\frac{1}{k} - \sum_{p=2}^{n+1}\frac{1}{p} + \frac{1}{2}\sum_{q=3}^{n+2}\frac{1}{q} \qquad \text{en ayant posé } p = k+1 \text{ et } q = k+2 \\
&= \frac{1}{2}\left(1 + \frac{1}{2} + \sum_{k=3}^{n}\frac{1}{k}\right) - \left(\frac{1}{2} + \sum_{p=3}^{n}\frac{1}{p} + \frac{1}{n+1}\right) + \frac{1}{2}\left(\sum_{q=2}^{n}\frac{1}{q} + \frac{1}{n+1} + \frac{1}{n+2}\right) \\
&= \boxed{\frac{1}{4} - \frac{1}{2(n+1)(n+2)}}
\end{aligned}$$

Correction 3.

1.

$$\begin{aligned}
\cos(2\tfrac{\pi}{8}) &= 2\cos^2(\tfrac{\pi}{8}) - 1 \\
\tfrac{\sqrt{2}}{2} &= 2\cos^2(\tfrac{\pi}{8}) - 1 \\
1 + \tfrac{\sqrt{2}}{2} &= 2\cos^2(\tfrac{\pi}{8}) \\
\tfrac{2+\sqrt{2}}{4} &= \cos^2(\tfrac{\pi}{8})
\end{aligned}$$

Or $\cos(\frac{\pi}{8}) > 0$ car $\frac{\pi}{8} \in [0, \frac{\pi}{2}[$ donc $\boxed{\cos(\frac{\pi}{8}) = \frac{\sqrt{2+\sqrt{2}}}{2}}$.

Même principe : $\cos^2(2\frac{\pi}{8}) = 1 - 2\sin^2(\frac{\pi}{8})$ et $\sin(\frac{\pi}{8}) > 0$.

On trouve $\boxed{\sin(\frac{\pi}{8}) = \frac{\sqrt{2-\sqrt{2}}}{2}}$.

2. Donc $\left(\sqrt{2+\sqrt{2}} + i\sqrt{2-\sqrt{2}}\right)^8 = \left(2(\cos(\frac{\pi}{8}) + i\sin(\frac{\pi}{8}))\right)^8 = \left(2e^{i\frac{\pi}{8}}\right)^8 = 2^8 e^{i\frac{\pi}{8}\times 8} = -2^8 = \boxed{-256}$.